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A SIMPLE PROOF OF THE SPEARMAN-BROWN FORMULA FOR CONTINUOUS TEST LENGTHS

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A direct proof is given for the generalized Spearman-Brown formula for any real multiple of test length.

For a generalized version of classical test theory, Lord and Novick [1968, Chapter 5] proved the well-known Spearman-Brown formula for the reliability of a lengthened test. In this generalized model proposed by Woodbury [1963], the test score random variable $X(t)$ is generated by a weakly-stationary stochastic process of length t, where t may take on any positive, real value. Nevertheless, in their proof, Lord and Novick assumed that t was a positive integer. They then argued that the proof also applied to non-integral, rational values of t by a redefinition of unit length, and to irrational values of t by a limiting argument.

Here I present a simple, more direct proof of the formula for all positive, real values of t. I shall use the notation, some of the earlier results, and the reference numbers of Lord and Novick [1968]. The assumptions of the model are stated in section 5.3 of their work.

Let X and X' be parallel measures of unit length ($t = 1$), and $X(k)$ and $X'(k)$ be corresponding measures of length $kt = k$. The theorem (5.10.1) states that the reliability of the lengthened (or shortened) test $X(k)$ can be written as

$$\rho\left(\frac{X(k)}{k}, \frac{X'(k)}{k}\right) = \frac{k\rho_{XX'}}{1 + (k-1)\rho_{XX'}}.$$

Proof

In (5.8.1b), Lord and Novick give a general formula for the correlation between observed-score random variables from the same process:

$$\rho\left(\frac{X(t_2) - X(t_1)}{t_2 - t_1}, \frac{X(t_4) - X(t_3)}{t_4 - t_3}\right) = \frac{\sigma_T^2 + \dfrac{a^2 t^*}{(t_2 - t_1)(t_4 - t_3)}}{\left(\sigma_T^2 + \dfrac{a^2}{t_2 - t_1}\right)^{1/2}\left(\sigma_T^2 + \dfrac{a^2}{t_4 - t_3}\right)^{1/2}},$$

where T is the true-score random variable, t_i ($i = 1, \cdots, 4$) refers to points on the t scale, $t^* = (t_1, t_2) \cap (t_3, t_4)$, and a^2 is the error variance for a test

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of unit length. In the case of parallel measures, $t_4 - t_3 = t_2 - t_1 = k$ and $t^* = 0$ (Theorem 5.3.1). Then (5.8.1b) reduces to

$$\rho\left(\frac{X(k)}{k}, \frac{X'(k)}{k}\right) = \frac{\sigma_T^2}{\sigma_T^2 + a^2/k},$$

where $X(k) = X(t_2) - X(t_1)$ and $X'(k) = X(t_4) - X(t_3)$. Note from (5.8.1a) that this is also the squared correlation between true and observed scores. For parallel measures of unit length, this further reduces to

$$\rho_{XX'} = \frac{\sigma_T^2}{\sigma_T^2 + a^2}.$$

Then

$$\begin{aligned}\rho\left(\frac{X(k)}{k}, \frac{X'(k)}{k}\right) &= \frac{k\sigma_T^2}{k\sigma_T^2 + a^2} \\ &= \frac{k\sigma_T^2}{k\sigma_T^2 - \sigma_T^2 + \sigma_T^2 + a^2} \\ &= \frac{k\sigma_T^2}{(k-1)\sigma_T^2 + (\sigma_T^2 + a^2)} \\ &= \frac{k\left(\dfrac{\sigma_T^2}{\sigma_T^2 + a^2}\right)}{(k-1)\left(\dfrac{\sigma_T^2}{\sigma_T^2 + a^2}\right) + 1} \\ &= \frac{k\rho_{XX'}}{(k-1)\rho_{XX'} + 1},\end{aligned}$$

which completes the proof.

In similar fashion, one can readily prove theorem (5.9.1b),

$$\sigma^2\left(\frac{X(k)}{k}\right) = \frac{\sigma_X^2}{k}\,[1 + (k-1)\rho_{XX'}],$$

without restricting the application to integral values of k.

REFERENCES

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